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Path-integral methods for turbulent diffusion

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We derive a path-integral representation for the effective diffusion function of a passive scalar field. We use it to calculate the long-time effective diffusivity in Gaussian turbulence in the near-Markovian limit. Our results confirm the negative effect of vorticity predicted by previous discussions. They also demonstrate that the helicity of the turbulence when present may be as important an influence as the vorticity.

1. Introduction

In this paper we show how path-integral techniques may be used to study the problem of turbulent diffusion. Since the underlying problem is one involving the statistical distribution of paths it seems natural and appropriate to approach it in this way. As their use in quantum mechanics (Feynman & Hibbs 1965), random media (Flatte 1979) and diffusion problems (Onsager & Machlup 1953; Graham 1977) has shown, path integrals not only provide considerable insight but are often effective and convenient computational tools as well.

The plan of the paper is as follows. First we introduce, in an appropriate form, the path integral for a passive scalar field subject to molecular diffusion and convection in a given velocity field. We then obtain a representation for the effective diffusion function (EDF) for a turbulent flow by averaging over an appropriate ensemble of velocity fields.

As an application of the representation we study the effective long-time diffusivity for Gaussian turbulence in the near-Markovian limit. (Kraichnan 1968; Phythian & Curtiss 1978; Knobloch 1977, 1980). Although this limit does not correspond directly to physical turbulence the results are nevertheless of interest. In particular they exhibit the effect of helicity on the turbulent dispersion.

2. Path-integral solution of the diffusion equation

The equation that governs the evolution of a temperature distribution or chemical concentration $\theta(x,t)$ subject to diffusion and convection by an incompressible fluid is

$$\frac{\partial \theta}{\partial t}+\mathbf{u}\,.\,\nabla\theta = \kappa\nabla^2\theta, \tag{2.1}$$

where the velocity field of the fluid is $\mathbf{u} \equiv \mathbf{u}(\mathbf{x}, t)$ and the molecular diffusivity is κ. We have also

$$\nabla\,.\,\mathbf{u} = 0. \tag{2.2}$$

Equation (2.1) has the following path-integral solution (Onsager & Machlup 1953; Feynman & Hibbs 1965; Graham 1977):

$$\theta = G(\mathbf{x}t|\mathbf{x}'t') \equiv \int d[\mathbf{x}] \exp\left\{-\frac{1}{4\kappa}\int_{t'}^{t} d\tau[\dot{\mathbf{x}}(\tau)-\mathbf{u}(\mathbf{x}(\tau),\tau)]^2\right\}, \tag{2.3}$$

which satisfies

$$G(\mathbf{x},t'|\mathbf{x}',t') = \delta(\mathbf{x}-\mathbf{x}'). \tag{2.4}$$

In (2.3) $d[\mathbf{x}]$ stands for the measure on the set of paths $\{\mathbf{x}(\tau)\}$ over which we integrate. These paths satisfy $\mathbf{x}(t') = \mathbf{x}'$ and $\mathbf{x}(t) = \mathbf{x}$.

It is useful both for clarity and subsequent development to recall the definition of the path integral. We select a sequence of times $\{t_0 \equiv t', t_1, \ldots, t_n, t_{n+1} \equiv t\}$, where

$$t_l = t' + l\epsilon \quad (l = 0, 1, \ldots, n+1),$$

and $\epsilon = (t-t')/(n+1)$. The path $\mathbf{x}(\tau)$ is now approximated by a sequence of points $\{\mathbf{x}_0 \equiv \mathbf{x}', \mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{x}_{n+1} \equiv \mathbf{x}\}$ and the exponent in the integrand of (2.3) is approximated by the sum

$$S_n = \sum_{l=0}^{n} \epsilon\left[\frac{\mathbf{x}_{l+1}-\mathbf{x}_l}{\epsilon} - \mathbf{u}_l\right]^2, \tag{2.5}$$

where

$$\mathbf{u}_l = \mathbf{u}(\xi_l, t_l), \tag{2.6}$$

and ξ_l is chosen to lie somewhere on the line joining $\mathbf{x}_l$ to $\mathbf{x}_{l+1}$. In the present case for which $\nabla.\mathbf{u} \equiv 0$ the precise method for choosing ξ_l does not influence the result, although in general it will. In the case of compressible flow, then, we would have to exercise more care in using the path integral.

The measure over the paths is approximated by (Feynman & Hibbs 1965)

$$d[\mathbf{x}] = \frac{1}{A}\prod_{l=1}^{n}\left(\frac{d^3\mathbf{x}_l}{A}\right), \tag{2.7}$$

where

$$A = (4\pi\kappa\epsilon)^{\frac{3}{2}}. \tag{2.8}$$

Equation (2.3), then, is to be interpreted as meaning

$$G(\mathbf{x}t|\mathbf{x}'t') = \lim_{n\to\infty}\frac{1}{A}\int\prod_{l=1}^{n}\left(\frac{d^3\mathbf{x}_l}{A}\right)\exp\left\{-\frac{1}{4\kappa}S_n\right\}. \tag{2.9}$$

When the fluid is at rest and $\mathbf{u}$ vanishes the integral in (2.9) is easily evaluated, and yields the standard result

$$G = G_0(\mathbf{x}t|\mathbf{x}'t') \equiv (4\pi\kappa(t-t'))^{-\frac{3}{2}}\exp\left\{-\frac{(\mathbf{x}-\mathbf{x}')^2}{4\kappa(t-t')}\right\}. \tag{2.10}$$

Although the path-integral representation in (2.3) is already quite useful it suffers from the disadvantage that some of the κ-dependence is buried in the measure $d[\mathbf{x}]$. It is more convenient therefore to introduce a modified representation in which the κ-dependence is completely explicit. This new representation has the additional advantage that the $\mathbf{u}$-dependence of the exponent in the integrand is linear rather than quadratic.

To derive this new representation we introduce a sequence of variables $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n$ and make use of the identity

$$(4\pi\kappa\epsilon)^{-\frac{3}{2}}\exp\left\{-\frac{\epsilon}{4\kappa}\left(\frac{\mathbf{x}_{l+1}-\mathbf{x}_l}{\epsilon}-\mathbf{u}_l\right)^2\right\} = \int\frac{d^3\mathbf{p}_l}{(2\pi)^3}\exp\left\{-\epsilon\kappa\mathbf{p}_l^2 + i\epsilon\mathbf{p}_l\cdot\left(\frac{\mathbf{x}_{l+1}-\mathbf{x}_l}{\epsilon}-\mathbf{u}_l\right)\right\}. \tag{2.11}$$

Combining (2.11) with (2.5) and (2.9) we find that

$$G(\mathbf{x}t|\mathbf{x}'t') = \lim_{n\to\infty} \int \prod_{l=1}^{n} d^3\mathbf{x}_l \prod_{l=0}^{n} \frac{d^3\mathbf{p}_l}{(2\pi)^3} \exp\left\{-\sum_{l=0}^{n} \epsilon\left[\kappa\mathbf{p}_l^2 - i\mathbf{p}_l\cdot\left(\frac{\mathbf{x}_{l+1}-\mathbf{x}_l}{\epsilon}-\mathbf{u}_l\right)\right]\right\}. \quad (2.12)$$

This limit can now be interpreted as a double path integral, with the result

$$G(\mathbf{x}t|\mathbf{x}'t') = \int d[\mathbf{x}]\, d[\mathbf{p}] \exp\left\{-\int_{t'}^{t} d\tau[\kappa\mathbf{p}^2(\tau) - i\mathbf{p}(\tau)\,.\,(\dot{\mathbf{x}}(\tau)-\mathbf{u}(\mathbf{x}(\tau),\tau))]\right\}, \quad (2.13)$$

where now the measures are

$$d[\mathbf{x}] \sim \prod_{l=1}^{n} d^3\mathbf{x}_l, \quad (2.14)$$

$$d[\mathbf{p}] \sim \prod_{l=0}^{n} \frac{d^3\mathbf{p}_l}{(2\pi)^3}. \quad (2.15)$$

The $\mathbf{x}$-integral is over the same paths as before, while the $\mathbf{p}$-integral is over all paths in $\mathbf{p}$-space defined on the range $t' \leqslant \tau \leqslant t$ with no restriction at the end points. It is interesting to note that the representation for the EDF in (2.13) bears the same relation to the original one in (2.3) as the canonical path integral in quantum mechanics bears to the Lagrangian path integral of Feynman & Hibbs (1965).

3. The effective diffusion function

The effective diffusion function (EDF) is given by

$$\mathscr{G}(\mathbf{x},t|\mathbf{x}',t') = \langle G(\mathbf{x},t|\mathbf{x}',t')\rangle, \quad (3.1)$$

where $\langle A\rangle$ represents the expectation value of A over the ensemble of velocity fields representing the turbulence. Equation (2.13) then yields the following path-integral representation for $\mathscr{G}$:

$$\mathscr{G}(\mathbf{x},t|\mathbf{x}',t') = \int d[\mathbf{x}]\, d[\mathbf{p}] \exp\left\{-\int_{t'}^{t} d\tau[\kappa\mathbf{p}^2(\tau) - i\mathbf{p}(\tau)\,.\,\dot{\mathbf{x}}(\tau)]\right\} \times \left\langle \exp\left\{-i\int_{t'}^{t} d\tau\, \mathbf{p}(\tau)\,.\,\mathbf{u}(\mathbf{x}(\tau),\tau)\right\}\right\rangle. \quad (3.2)$$

The expectation value in (3.2) is just a special case of the generating functional for the Eulerian correlation functions of the velocity field. We can recover the standard perturbation theory for (2.1) by expanding the exponential in powers of the velocity field.

In the case of Gaussian turbulence we can evaluate the expectation value in terms of the two-point correlation function (assuming $\langle\mathbf{u}\rangle = 0$) and obtain the result

$$\mathscr{G}(\mathbf{x},t|\mathbf{x}',t') = \int d[\mathbf{x}]\, d[\mathbf{p}] \exp\left\{-\int_{t'}^{t} d\tau[\kappa\mathbf{p}^2(\tau) - i\mathbf{p}(\tau)\,.\,\dot{\mathbf{x}}(\tau)]\right\} \times \exp\left\{-\frac{1}{2}\int_{t'}^{t} \delta\tau\, d\tau'\, p_i(\tau)\, \Delta_{ij}(\tau|\tau')\, p_j(\tau')\right\}, \quad (3.3)$$

where

$$\Delta_{ij}(\tau|\tau') = \langle u_i(\mathbf{x}(\tau),\tau)\, u_j(\mathbf{x}(\tau'),\tau')\rangle. \quad (3.4)$$

Even in the case of Gaussian turbulence, then, the evaluation of $\mathscr{G}$ is a non-trivial problem because of the path dependence of Δ_{ij}.

One circumstance in which this dependence is suppressed is the Markovian limit, in which we can make the approximation

$$\langle u_i(\mathbf{x},t)\, u_j(\mathbf{x}',t')\rangle = D_{ij}(\mathbf{x}-\mathbf{x}')\,\delta(t-t'), \tag{3.5}$$

where

$$D_{ij}(\mathbf{x}-\mathbf{x}') = \int_{-\infty}^{\infty} d\tau\, \langle u_i(\mathbf{x},\tau)\, u_j(\mathbf{x}',0)\rangle. \tag{3.6}$$

We have assumed homogeneity and time independence for the turbulence.

It follows that

$$\Delta_{ij}(\tau\,|\,\tau') = D_{ij}(\mathbf{x}(\tau)-\mathbf{x}(\tau'))\,\delta(\tau-\tau'). \tag{3.7}$$

That is

$$\Delta_{ij}(\tau\,|\,\tau') = D_{ij}(0)\,\delta(\tau-\tau'). \tag{3.8}$$

If we assume isotropy for the turbulence then

$$D_{ij}(0) = \tfrac{1}{3}\delta_{ij}\,D, \tag{3.9}$$

where

$$D = D_{ii}(0) = \int_{-\infty}^{\infty} d\tau\,\langle \mathbf{u}(0,\tau)\,.\,\mathbf{u}(0,0)\rangle. \tag{3.10}$$

We find that the path-integral representation for the EDF takes the form

$$\mathscr{G}(\mathbf{x},t|\mathbf{x}',t') = \int d[\mathbf{x}]\,d[\mathbf{p}]\,\exp\left\{-(\kappa+\tfrac{1}{6}D)\int_{t'}^{t} d\tau\,\mathbf{p}^2(\tau)+i\int_{t'}^{t} d\tau\,\mathbf{p}(\tau)\,.\,\dot{\mathbf{x}}(\tau)\right\}. \tag{3.11}$$

The EDF is then just a simple diffusion function with an effective diffusivity of $\kappa+\frac{1}{6}D$.

As has been pointed out (Kraichnan 1968; Phythian & Curtiss 1978) the Markovian limit imposes an unrealistic constraint on the turbulence in which the velocity-correlation time is very much smaller than the eddy-circulation time. It is important therefore to consider corrections to the Markovian limit. We calculate these to lowest order in §4.

4. Corrections to the Markovian limit

Corrections to the Markovian limit are obtained by first expanding the correlation function as a Taylor series in position. That is

$$\langle u_i(\mathbf{x},\,\tau)\,u_j(\mathbf{x}',\,\tau')\rangle = \tfrac{1}{3}\delta_{ij}\langle \mathbf{u}(0,\tau)\,.\,\mathbf{u}(0,\tau')\rangle$$
$$+\Delta_{ij,k}(\tau|\tau')\,(\mathbf{x}-\mathbf{x}')_k+\tfrac{1}{2}\Delta_{ij,kl}(\tau|\tau')\,(\mathbf{x}-\mathbf{x}')_k\,(\mathbf{x}-\mathbf{x}')_l+\ldots. \tag{4.1}$$

In writing the first term in this expansion we have used the assumed isotropy of the turbulence. We also have

$$\Delta_{ij,k}(\tau|\tau') = \tfrac{1}{6}h(\tau|\tau')\,\epsilon_{ijk}, \tag{4.2}$$

where

$$h(\tau|\tau') = \langle \boldsymbol{\omega}(0,\tau)\,.\,\mathbf{u}(0,\tau')\rangle, \tag{4.3}$$

$\boldsymbol{\omega}$ being the vorticity of the fluid, and

$$\Delta_{ij,kl}(\tau|\tau') = \tfrac{1}{30}g(\tau|\tau')\,(\delta_{il}\delta_{jk}+\delta_{ik}\delta_{jl}-4\delta_{ij}\delta_{kl}), \tag{4.4}$$

with

$$g(\tau|\tau') = \langle \boldsymbol{\omega}(0,\tau)\,.\,\boldsymbol{\omega}(0,\tau')\rangle. \tag{4.5}$$

Our approximation procedure is to treat the first term of the expansion in (4.1) exactly and the remaining terms perturbatively. The justification for this is the assumption that we are near the Markovian limit, in which the effect of these remaining terms disappears.

The expression for the EDF is then

$$\mathscr{G}(\mathbf{x},t|\mathbf{x}',t) = \int d[\mathbf{x}]\,d[\mathbf{p}]\exp\left\{-\frac{1}{2}\int d\tau d\tau'\mathbf{p}(\tau)\,.\,\mathbf{p}(\tau')f(\tau|\tau')+i\int_{t'}^{t}d\tau\,\mathbf{p}(\tau)\,.\,\dot{\mathbf{x}}(\tau)\right\}$$
$$\times\left[1-\frac{1}{2}\int_{t'}^{t}d\tau d\tau'\Delta_{ij,k}(\tau|\tau')\,p_i(\tau)\,p_j(\tau')\,(\mathbf{x}(\tau)-\mathbf{x}(\tau'))_k\right.$$
$$-\frac{1}{4}\int_{t'}^{t}d\tau d\tau'\Delta_{ij,kl}(\tau|\tau')\,p_i(\tau)\,p_j(\tau')\,(\mathbf{x}(\tau)-\mathbf{x}(\tau'))_k\,(\mathbf{x}(\tau)-\mathbf{x}(\tau'))_l$$
$$\left.+\frac{1}{8}\left(\int_{t'}^{t}d\tau d\tau'\Delta_{ij,k}(\tau/\tau')\,p_i(\tau)\,p_j(\tau')\,(\mathbf{x}(\tau)-\mathbf{x}(\tau'))_k\right)^2+\ldots\right], \tag{4.6}$$

where

$$f(\tau|\tau') = 6\kappa\delta(\tau-\tau')+\langle\mathbf{u}(0,\tau)\,.\,\mathbf{u}(0,\tau')\rangle. \tag{4.7}$$

We have truncated the expansion so as to retain all those terms that involve up to two spatial derivatives of the correlation function.

Considerations of reflection symmetry tell us without further calculation that the first term, which contains only a single power of $\Delta_{ij,k}$, will not contribute to the result. The next two terms provide the first non-vanishing corrections. We will refer to them as respectively the vorticity and helicity corrections.

The various terms in the expansion for the EDF can be obtained from the functional

$$I[\mathbf{v},\mathbf{q}] = \int d[\mathbf{x}]\,d[\mathbf{p}]\exp\left\{-\frac{1}{6}\int_{t'}^{t}d\tau d\tau'\mathbf{p}(\tau)\,.\,\mathbf{p}(\tau')f(\tau|\tau')\right\}$$
$$\times\exp\left\{i\int_{t'}^{t}d\tau[\mathbf{p}(\tau)\,.\,\dot{\mathbf{x}}(\tau)-\mathbf{p}(\tau)\,.\,\mathbf{v}(\tau)-\mathbf{q}(\tau)\,.\,\dot{\mathbf{x}}(\tau)]\right\}, \tag{4.8}$$

which is evaluated in the appendix. For example the lowest approximation is

$$\mathscr{G}_0(\mathbf{x},t|\mathbf{x}',t') = I[0,0].$$

Referring to the appendix we see that this is

$$\mathscr{G}_0(\mathbf{x},t|\mathbf{x}',t') = \frac{1}{(2\pi)^{\frac{3}{2}}\sigma_0^3}\exp\left\{-\frac{(\mathbf{x}-\mathbf{x}')^2}{2\sigma_0^2}\right\}, \tag{4.9}$$

where

$$\sigma_0^2 = \frac{1}{3}\int_{t'}^{t}d\tau d\tau' f(\tau|\tau'). \tag{4.10}$$

The vorticity correction can be expressed as

$$\delta\mathscr{G}_\omega = -\frac{1}{4}\int_{t'}^{t}d\tau d\tau'\Delta_{ij,kl}(\tau|\tau')\int_{\tau'}^{\tau}d\tau'' d\tau'''\,I_{ijkl}, \tag{4.11}$$

where

$$I_{ijkl} = \frac{\delta^4}{\delta v_i(\tau)\,\delta v_j(\tau')\,\delta q_k(\tau'')\,\delta q_l(\tau''')}I\bigg|_{\mathbf{v}=\mathbf{q}=0} \tag{4.12}$$

Ultimately this may be expressed as

$$\delta\mathscr{G}_\omega = \frac{1}{120}\mathscr{G}_0\int_{t'}^{t}d\tau d\tau'\int_{\tau'}^{\tau}d\tau'' d\tau'''\,g(\tau|\tau')\left\{-\frac{2}{\sigma_0^2}\mathrm{F}(\tau'')\,F(\tau''')\,[(\mathbf{x}-\mathbf{x}')^2]^2\right.$$
$$\left.+\frac{10}{\sigma_0^6}\left[2F(\tau'')\,F(\tau''')-\frac{1}{3}\sigma_0^2 f(\tau''|\tau''')\right](\mathbf{x}-\mathbf{x}')^2-\frac{30}{\sigma_0^4}\left[F(\tau'')\,F(\tau''')-\frac{1}{3}\sigma_0^2 f(\tau''|\tau''')\right]\right\}, \tag{4.13}$$

where

$$F(\tau'') = \frac{1}{3}\int_{t'}^{t} d\tau f(\tau|\tau''). \tag{4.14}$$

The helicity correction can be expressed as

$$\delta\mathscr{G}_h = \frac{1}{8}\int_{t'}^{t} d\tau_1 d\tau_1' \int_{\tau_1'}^{\tau_1} d\tau_1'' \int_{t'}^{t} d\tau_2 d\tau_2' \int_{\tau_2'}^{\tau_2} d\tau_2'' \Delta_{ij,k}(\tau_1|\tau_1')\, \Delta_{lm,n}(\tau_2|\tau_2')\, I_{ijklmn}, \tag{4.15}$$

where

$$I_{ijklmn} = -\frac{\delta^6}{\delta v_i(\tau_1)\, \delta v_j(\tau_1')\, \delta q_k(\tau_1'')\, \delta v_l(\tau_2)\, \delta v_m(\tau_2')\, \delta q_n(\tau_2'')} I \bigg|_{V=\mathbf{q}=0}. \tag{4.16}$$

Ultimately this can be put in the form

$$\delta\mathscr{G}_h = \frac{1}{18\sigma_0^4} N\mathscr{G}_0[(\mathbf{x}-\mathbf{x}')^2 - 3\sigma_0^2], \tag{4.17}$$

where

$$N = \int_{t'}^{t} d\tau_1 \int_{t'}^{\tau_1} d\tau_2 \int_{\tau_2}^{\tau_1} d\tau_2' \int_{\tau_2}^{\tau_2'} d\tau_1'\, h(\tau_1|\tau_1')\, h(\tau_2|\tau_2'). \tag{4.18}$$

It is easily verified that

$$\int d^3\mathbf{x}\, \delta\mathscr{G}_\omega = \int d^3\mathbf{x}\, \delta\mathscr{G}_h = 0, \tag{4.19}$$

so that two corrections do not alter the overall normalization of $\mathscr{G}$.

The lowest approximation to the dispersion is

$$\langle \mathbf{X}^2 \rangle_0 = 3\sigma_0^2, \tag{4.20}$$

and the lowest approximation to the fourth moment is

$$\langle (\mathbf{X}^2)^2 \rangle_0 = 15\sigma_0^4. \tag{4.21}$$

The effect of the vorticity correction is, for the dispersion,

$$\delta\langle \mathbf{X}^2 \rangle_\omega = -\frac{1}{6}\int_{t'}^{t} d\tau\, d\tau' \int_{\tau'}^{\tau} d\tau''\, d\tau'''\, g(\tau|\tau')\, f(\tau''|\tau'''), \tag{4.22}$$

and for the fourth moment,

$$\delta\langle (\mathbf{x}^2)^2 \rangle_\omega = -\frac{5}{3}\sigma_0^2 \int_{t'}^{t} d\tau\, d\tau' \int_{\tau'}^{\tau} d\tau''\, d\tau'''\, g(\tau|\tau')\, f(\tau''|\tau''') \\ -2\int_{t'}^{t} d\tau\, d\tau'\, g(\tau|\tau') \left[\int_{\tau'}^{\tau} d\tau''\, F(\tau'')\right]^2, \tag{4.23}$$

while the helicity correction yields

$$\delta\langle \mathbf{X}^2 \rangle_h = \tfrac{1}{3}N, \tag{4.24}$$

$$\delta\langle (\mathbf{X}^2)^2 \rangle = \tfrac{10}{3}\sigma_0^2 N. \tag{4.25}$$

The results imply that, within the approximation employed, the flatness factor R (Kraichnan 1970) is given by

$$R \equiv \frac{\langle (\mathbf{X}^2)^2 \rangle}{\langle \mathbf{X}^2 \rangle^2} = \frac{5}{3}\left(1 - \frac{2}{15\sigma_0^4}\int_{t'}^{t} d\tau\, d\tau'\, g(\tau|\tau') \left[\int_{\tau'}^{\tau} d\tau''\, F(\tau'')\right]^2\right). \tag{4.26}$$

Note that the helicity correction does not influence the behaviour of R.

At short times we find

$$\delta\langle \mathbf{X}^2 \rangle_\omega \approx -\tfrac{1}{3}\kappa\omega^2(t-t')^3 - \tfrac{1}{36}\,\omega^2 v^2 (t-t')^4 + \dots, \tag{4.27}$$

$$\delta\langle \mathbf{X}^2 \rangle_h \approx \tfrac{1}{72} h^2 (t-t')^4, \tag{4.28}$$

where $h = h(0|0)$, and

$$R \approx \tfrac{5}{3}(1-\tfrac{1}{45}\omega^2(t-t')^2) \tag{4.29}$$

The first term in (4.27) is consistent with the work of Saffman (1960), the terms $O((t-t')^4)$ can also be confirmed when $\kappa = 0$, by a Taylor expansion in time. Equation (4.29) is interesting because it is independent of the molecular diffusivity. Again it is easily verified for Gaussian turbulence when $\kappa = 0$.

At large times we have

$$\delta\langle \mathbf{X}^2\rangle_\omega \approx -\tfrac{1}{6}(t-t')\int_{-\infty}^{\infty} d\tau g(\tau|0)\int_0^{\tau} d\tau''\, d\tau'''\, f(\tau''|\tau'''), \tag{4.30}$$

$$\delta\langle \mathbf{X}^2\rangle_h \approx \tfrac{1}{3}(t-t')\int_{-\infty}^{\infty} d\tau \int_0^{\tau} d\tau' \int_0^{\tau'} d\tau''\, h(\tau|\tau'')\, h(0|\tau'), \tag{4.31}$$

$$R \approx \frac{5}{3}\left(1-\frac{T}{t-t'}\right), \tag{4.32}$$

where

$$T = \frac{2}{15}\int_{-\infty}^{\infty} d\tau\, \tau^2\, g(\tau|0). \tag{4.33}$$

Note that the vorticity correction continues to reduce the dispersion while the helicity correction remains positive.

All of these results are of course consistent with the original hypothesis of Taylor (1921) that at large times turbulent dispersion is controlled by an effective diffusivity.

5. A specific case and conclusions

Summarizing the results of the previous sections we find that the effective diffusivity controlling the long-time behaviour of the EDF is

$$\begin{aligned} K_{\text{eff}} = K + \frac{1}{6}\int_{-\infty}^{\infty} d\tau \langle \mathbf{u}(0,\tau)\,.\,\mathbf{u}(0,0)\rangle - \frac{1}{12}\int_{-\infty}^{\infty} d\tau\langle \boldsymbol{\omega}(0,\tau)\,.\,\boldsymbol{\omega}(0,0)\rangle\, \sigma_0^2(\tau) \\ + \frac{1}{18}\int_0^{\infty} d\tau \int_0^{\tau} d\tau' \int_0^{\tau'} d\tau'' \langle \boldsymbol{\omega}(0,\tau)\,.\,\mathbf{u}(0,\tau'')\rangle\, \langle \boldsymbol{\omega}(0,0)\,.\,\mathbf{u}(0,\tau')\rangle, \end{aligned} \tag{5.1}$$

where

$$\sigma_0^2(\tau) = 2\kappa|\tau| + \frac{1}{3}\int_0^{\tau} d\tau_1\, d\tau_2 \langle \mathbf{u}(0,\tau_1)\,.\,\mathbf{u}(0,\tau_2)\rangle. \tag{5.2}$$

It is interesting to evaluate these results in a simple model. For simplicity we assume all correlation functions have the same time dependence, namely

$$\begin{aligned} \langle \mathbf{u}(0,\tau)\,.\,\mathbf{u}(0,\tau')\rangle &= v^2 e^{-\Omega|\tau-\tau'|}, \\ \langle \boldsymbol{\omega}(0,\tau)\,.\,\boldsymbol{\omega}(0,\tau')\rangle &= \omega^2 e^{-\Omega|\tau-\tau'|}, \\ \langle \boldsymbol{\omega}(0,\tau)\,.\,\mathbf{u}(0,\tau')\rangle &= \alpha\omega v e^{-\Omega|\tau-\tau'|}. \end{aligned} \tag{5.3}$$

For this model the velocity correlation time is $\tau_v = \Omega^{-1}$ and the eddy circulation time is $\tau_e = \omega^{-1}$. Hence the ratio $\tau_v/\tau_e = \omega/\Omega$, so we are concerned with the limit of small vorticity. We readily find then that, for $\tau > 0$,

$$\sigma_0^2(\tau) = 2\left(\kappa + \frac{v^2}{3\Omega}\right)\tau - \frac{2v^2}{3\Omega}(1-e^{-\Omega\tau}), \tag{5.4}$$

and that

$$\kappa_{\text{eff}} = \kappa\left(1-\frac{1}{3}\frac{\omega^2}{\Omega^2}\right)+\frac{v^2}{3\Omega}\left(1-\tfrac{1}{12}(2-\alpha^2)\frac{\omega^2}{\Omega^2}\right). \tag{5.5}$$

From (4.33) we see also that the characteristic timescale T, in which the EDF returns to a Gaussian shape, is given by

$$T = \frac{8}{15}\frac{\omega^2}{\Omega^2}\frac{1}{\Omega}, \tag{5.6}$$

that is

$$T = \frac{8}{15}\left(\frac{\tau_v}{\tau_e}\right)^2 \tau_v. \tag{5.7}$$

In this model then, the EDF evolves towards a standard Gaussian diffusion function in three stages. The first stage occupies a time interval $0 < t-t' \leqslant T \ll \tau_v$. During this stage R lies below its standard value of $\frac{5}{3}$, indicating that the EDF is more sharply peaked than a Gaussian shape with the same dispersion. The initial decrease of R from $\frac{5}{3}$ is proportional to $\omega^2(t-t')^2$. Towards the end of this stage R increases again towards $\frac{5}{3}$ and the EDF attains a Gaussian shape. During this increase the departure of R from its limiting value is proportional to $T/(t-t')$.

The second stage lies in the time interval $T < t-t' \lesssim \tau_v$. During this stage, while the EDF has a Gaussian shape, the dispersion evolves in time in a way that reflects the details of the time dependence of the velocity correlation function.

The third stage occupies the time interval $\tau_v < t-t' < \infty$. Now the EDF is not only Gaussian in shape but also the dispersion increases linearly with time. In this stage then the EDF satisfies the standard diffusion equation

$$\frac{\partial}{\partial t}\mathscr{G} = \kappa_{\text{eff}}\,\nabla^2\mathscr{G} \tag{5.8}$$

with κ_{eff} given by (7.5).

Clearly for more realistic values of τ_v/τ_e we have $T \sim \tau_v$, so the above picture must be at least partly modified. The first stage for example will clearly overlap and will not be readily distinguishable from the second stage. However, it seems plausible that certain qualitative features will remain. These are the following.

(i) The initial time dependence of R. This can in any case be established quite generally.

(ii) The final time dependence of R. It is highly likely that the estimate of T will be affected by higher corrections. However, it is difficult to see how the fact that

$$\tfrac{5}{3}-R = O((t-t')^{-1})$$

will be altered. The significant point is that this quantity has an inverse rather than for example, an exponential dependence on time.

(iii) The fact that the helicity when present may be just as important as the vorticity in determining the effective long-time diffusivity. This possibility has not been generally emphasised, although similar squared helicity effects have been discussed for magnetic and scalar fields in the context of numerical calculations (Kraichnan 1976, 1977). There too the effect of helicity on the effective diffusivity is positive. Equation (7.5) indicates that while the helicity modifies the lower-order estimate of eddy diffusivity it does not interact with the molecular diffusivity as does the vorticity. It is therefore of particular importance when molecular diffusivity is low or absent.

(iv) The fact that the expressions for the long-time diffusivity in (7.1) and (7.5) are entirely in line with previous arguments (Saffman 1960) and calculations

(Kraichnan 1970, Phythian & Curtis 1978) in that they suggest that the interaction of molecular diffusivity and vorticity is such as to provide a negative contribution to κ_{eff}.

Finally, on the path-integral technique itself we would like to remark that we feel we have shown that it can be a useful way of approaching the problem of turbulent diffusion. Insofar as it is used, as here, to derive a perturbation series for the quantities of interest it could be circumvented by other, perhaps more familiar, techniques. However, the path-integral representation for the EDF is very flexible, and can provide a natural starting point for many different kinds of calculation.

Appendix

We evaluate the moment-generating functional used to calculate terms in the perturbation series for $\mathscr{G}(\mathbf{x}, t|\mathbf{x}', t')$. It is

$$I[\mathbf{v},\mathbf{q}] = \int d[\mathbf{x}]d[\mathbf{p}] \exp\left\{-\frac{1}{6}\int_{t'}^{t} d\tau d\tau'\, \mathbf{p}(\tau).\mathbf{p}(\tau') f(\tau|\tau')\right\} \times \exp\left\{i\int_{t'}^{t} d\tau[\mathbf{p}(\tau).\dot{\mathbf{x}}(\tau)-\mathbf{p}(\tau).\mathbf{v}(\tau)-\mathbf{q}(\tau).\dot{\mathbf{x}}(\tau)]\right\}. \quad \text{(A 1)}$$

Now introduce a randomly fluctuating vector $\mathbf{w}(\tau)$, which obeys Gaussian statistics and has a correlation function

$$\langle w_\alpha(\tau)\, w_\beta(\tau')\rangle = \tfrac{1}{3}\delta_{\alpha\beta} f(\tau|\tau'). \quad \text{(A 2)}$$

We have

$$\exp\left\{-\frac{1}{6}\int_{t'}^{t} d\tau d\tau' \mathbf{p}(\tau).\ \mathbf{p}(\tau') f(\tau|\tau')\right\} = \left\langle \exp\left\{-i\int_{t'}^{t} d\tau\, \mathbf{p}(\tau).\omega(\tau)\right\}\right\rangle_{\mathbf{w}}. \quad \text{(A 3)}$$

It follows that

$$I[\mathbf{v},\mathbf{q}] = \left\langle \int d[\mathbf{x}]d[\mathbf{p}] \exp\left\{i\int_{t'}^{t} d\tau[\mathbf{p}(\tau).[\dot{\mathbf{x}}(\tau)-\mathbf{v}(\tau)-\mathbf{w}(\tau)]-\mathbf{q}(\tau).\dot{\mathbf{x}}(\tau)]\right\}\right\rangle_{\mathbf{w}} \quad \text{(A 4)}$$

If we shift the $\mathbf{p}$-integration variable so that

$$\mathbf{p}(\tau) \to \mathbf{p}(\tau)+\mathbf{q}(\tau), \quad d[\mathbf{p}] \to d[\mathbf{p}] \quad \text{(A 5)}$$

then we find

$$I[\mathbf{v},\mathbf{q}] = \left\langle \int d[\mathbf{x}]d[\mathbf{p}] \exp\left\{i\int_{t'}^{t} d\tau\, \mathbf{p}(\tau).[\dot{\mathbf{x}}(\tau)-\mathbf{v}(\tau)-\mathbf{w}(\tau)]\right\} \times \exp\left\{-i\int_{t'}^{t} d\tau\, \mathbf{q}(\tau).(\mathbf{v}(\tau)+\mathbf{w}(\tau))\right\}\right\rangle_{\mathbf{w}}. \quad \text{(A 6)}$$

That is

$$I[\mathbf{v},\mathbf{q}] = \left\langle \delta(\mathbf{x}-\mathbf{x}'-\mathbf{R}-\boldsymbol{\rho}) \exp\left\{-i\int_{t'}^{t} d\tau\, \mathbf{q}(\tau).[\mathbf{v}(\tau)+\mathbf{w}(\tau)]\right\}\right\rangle_{\mathbf{w}}, \quad \text{(A 7)}$$

where

$$\mathbf{R} = \int_{t'}^{t} d\tau\, \mathbf{v}(\tau), \quad \boldsymbol{\rho}(\tau) = \int_{t'}^{t} d\tau\, \mathbf{w}(\tau).$$

Now introduce the standard representation for the δ-function so that

$$I[\mathbf{v},\mathbf{q}] = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \exp\{i\mathbf{k}.(\mathbf{x}-\mathbf{x}')\} \left\langle \exp\left\{-i\int_{t'}^{t} d\tau[\mathbf{q}(\tau)+\mathbf{k}].\mathbf{w}(\tau)\right\}\right\rangle_{\mathbf{w}} \times \exp\left\{-i\int_{t'}^{t} d\tau\, \mathbf{q}(\tau).\mathbf{w}(\tau)\right\}. \quad \text{(A 8)}$$

Setting

$$\left.\begin{aligned}\sigma_0^2 &= \frac{1}{3}\int_{t'}^{t} d\tau\, d\tau'\, f(\tau|\tau') \\ F(\tau') &= \frac{1}{3}\int_{t'}^{t} d\tau\, f(\tau|\tau') \\ \mathbf{Q} &= \int_{t'}^{t} d\tau'\, F(\tau')\, \mathbf{q}(\tau')\end{aligned}\right\} \qquad \text{(A 9)}$$

We find, using the property of Gaussian statistics, that

$$I[\mathbf{v},\mathbf{q}] = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \exp\{-\tfrac{1}{2}\sigma_0^2\mathbf{k}^2\} \exp\{i\mathbf{k}\,.\,(\mathbf{x}-\mathbf{x}'-\mathbf{R}+i\mathbf{Q})\} \times \exp\left\{-\tfrac{1}{6}\int_{t'}^{t} d\tau\, d\tau'\, \mathbf{q}(\tau)\,.\,\mathbf{q}(\tau')\, f(\tau|\tau') + i\int_{t'}^{t} d\tau\, \mathbf{q}(\tau)\,.\,\mathbf{v}(\tau)\right\}. \qquad \text{(A 10)}$$

On evaluating the Fourier transform we obtain finally

$$I[\mathbf{v},\mathbf{q}] = \frac{1}{(2\pi)^{\frac{3}{2}}\sigma_0^3} \exp\left\{-\frac{(\mathbf{x}-\mathbf{x}'-\mathbf{R}+i\mathbf{Q})^2}{2\sigma_0^2} - i\int_{t'}^{t} d\tau\, \mathbf{q}(\tau)\,.\,\mathbf{v}(\tau) - \frac{1}{6}\int d\tau\, d\tau'\, \mathbf{q}(\tau)\,.\,\mathbf{q}(\tau')\, f(\tau|\tau')\right\}. \qquad \text{(A 11)}$$

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